A SURVEY OF THE BGG DESCRIPTION OF COHOMOLOGY OF FLAG VARIETIES, WITH APPLICATIONS

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Abstract. In these lectures we will follow the original paper of Bernstein, Gel'fand, and Gel'fand (*Schubert cells and cohomology of the spaces G/P*) which identifies the cohomology of flag varieties W-equivariantly with certain (sub)quotients of explicit polynomial rings. We will develop the Lie combinatorics necessary to identify (and prove) "good" polynomial representatives with their Schubert class counterparts, along the way introducing the Bruhat order and the famous divided-difference operators. We will develop an "integration" formula by way of application, and I will explain how I used this in a crucial way for computer verification of two new instances of the Saturation Conjecture.

1. Lecture 1

1.1. **Notation.** G is a complex semsimple Lie group, assumed to be connected and simply connected. $H \subset B$ fixed maximal toral and Borel subgroups of G. $X = G/B$ the flag variety. N is the unipotent radical of B. $\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \mathfrak{n}$ are the respective Lie algebras. $\Phi \subset \mathfrak{h}^*$ is the root system (set of $\mathfrak{h}$-weights) of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$. The choice of B gives a set of positive roots $\Phi^+ \subset \Phi$; let $\Phi^- = -\Phi^+$ denote the set of negative roots. Recall the injective composite map

$$N_G(H)/H \hookrightarrow \mathrm{Aut}(H) \hookrightarrow \mathrm{Aut}(\mathfrak{h}) \simeq \mathrm{Aut}(\mathfrak{h}^*),$$

whose domain and image are both commonly known as the Weyl group W (the last isomorphism is the Killing form identification); see [FH04, Appendix D] for more details. For $\gamma \in \Phi^+$, $s_\gamma \in W$ is the reflection across the hyperplane orthogonal to $\gamma^\vee$ in $\mathfrak{h}^*$. $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ is the base of Φ determined by B (equiv. by Φ^+); here $r = \dim H$ is the rank of G. For ease of notation, s_i means s_{α_i} and is called a simple reflection; W is generated by $\{s_1, \ldots, s_r\}$. $\ell(\cdot) : W \to \mathbb{Z}_{\geqslant 0}$ is the length function:

$$\ell(w) = \min\{k \mid w = s_{i_1} \cdots s_{i_k}\}.$$

w_0 is the unique element of W of longest length.

To each $w \in W$, associate the Schubert cell (resp. variety) $C_w = B\dot{w} \subset G/B$ (resp. $X_w = \overline{B\dot{w}} \subset G/B$).

1.2. **Overview.** Recalling the Bruhat decomposition,

$$G = \bigsqcup_{w \in W} BwB,$$

we see that the C_w partition X. Each C_w is isomorphic to $\mathbb{C}^{\ell(w)}$ as an algebraic variety, and the X_w naturally give a cell decomposition of X. Therefore the fundamental classes $\mu(X_w) \in H_{2\ell(w)}(X)$ give a free $\mathbb{Z}$-basis of the singular homology of X. For more info, see [Spr98], [Ful96, Appendix B].

X has Poincaré duality, so the dual classes $[X_w] \in H^{2(\dim X - \ell(w))}(X)$ give a basis of the cohomology of X.

Question 1.1. *Remember $H^*(X)$ is a graded ring. Do we already know this ring (i.e., have a presentation of it)? Can we describe the basis elements $[X_w]$? Can we describe the W-action on it? Can we integrate forms on X?*

By freeness of $H^*(X)$ and the universal coefficient theorem, $H^*(X;\mathbb{Q}) \simeq H^*(X) \otimes \mathbb{Q}$. In what follows it will actually be simpler to keep $H^*(X;\mathbb{Q})$ in mind, and for ease of notation we will continue to use $H^*(X)$ for the rational cohomology.

Let $R = \mathbb{Q}[\alpha_i] = Sym^*(\mathfrak{h}^*_{\mathbb{Q}})$, where $\mathfrak{h}^*_{\mathbb{Q}}$ is the dual $\mathbb{Q}$-vector space to the subspace of $\mathfrak{h}$ generated over $\mathbb{Q}$ by the simple roots. Then there exists a map $R \to H^*(X)$ as follows. First, we describe a linear map $f : \mathfrak{h}^*_{\mathbb{Q}} \to H^2(X)$ as follows. To a weight $\omega : \mathfrak{h} \to \mathbb{C}$, we associate a 1-dimensional representation $\mathbb{C}_{-\omega}$ of B (N acts trivially). The diagonal quotient $G \times_B \mathbb{C}_{-\omega}$ is naturally the total space of a line bundle $\mathcal{L}_\omega$ over G/B. Set $f(\omega) = c_1(\mathcal{L}_\omega)$ to be the first Chern class of that line bundle. By a universal property, this extends to a $\mathbb{Q}$-algebra morphism $R \to H^*(X)$.

Theorem 1.2 ([Bor53],[AH61])**.** *$\theta : R/J \simeq H^*(X)$ is an isomorphism, where J is the ideal generated by W-invariant elements with no constant term. Furthermore, θ is W-equivariant.*

Remark 1.3. *In fact, the H-equivariant cohomology $H^*_H(X) \simeq R \otimes_{R^W} R$, so we should expect $R \otimes_{R^W} \mathbb{Q} \simeq R/J$ as rings, where $R^W \circlearrowleft \mathbb{Q}$ by multiplication by the constant term.*

Exercise 1.4. *Show that $R/J \simeq R \otimes_{R^W} \mathbb{Q}$ as rings.*

Here we explain the "W-equivariance" part of the theorem. Both rings in the above isomorphism have natural graded (grade R by 2·degree) W-actions. The action on R is clear (induced by the linear action on $\mathfrak{h}^*_{\mathbb{Q}}$). Topologically speaking (see [Han73]),

$$X \simeq K/T,$$

where K is a maximal compact subgroup of G ("compact form") and $T = K \cap H$ a maximal torus in K. The formula

$$w.\bar{k} := \overline{kw^{-1}}$$

indicates a left W-action on K/T which induces graded actions on $H_*(X)$ and $H^*(X)$ by functoriality.

The contribution of [BGG73] is to answer the second question above: that is, they (and we in these lectures) will construct the polynomials on the R/J side corresponding to the basis elements $[X_w]$. In cohomological degrees 0 and 2, we already have a headstart:

Lemma 1.5.

(1) $\theta(1) = [X_{w_0}]$
(2) $\theta(\omega_j) = [X_{w_0 s_j}]$ *for a fundamental weight* ω_j.

Proof. The first statement is clear by convention.

For the second statement, we calculate the divisor of zeros of a section of the line bundle $\mathcal{L}_{\omega_j}$; this will give an element in the Chow group $A^1(X)$, which we may identify with $H^2(X)$, and this is a valid method of calculating the Chern class (see [Har06, Appendix A.3]).

To that end, we remark that sections of $\mathcal{L}_{\omega_j}$ can be identified with algebraic functions $f : G \to \mathbb{C}$ satisfying

$$f(gb) = \omega_j(b)f(g)$$

for all $g \in G, b \in B$. **Exercise: show this.** Furthermore, there exists such a function f which also satisfies

$$f(ug) = f(g)$$

for all $u \in N$.

Let $x_i(t) = \exp(tX_{\alpha_i}), x_{-i}(t) = \exp(tX_{-\alpha_i})$ for any $i = 1, \ldots, r$, where the X_γs are a (certain) standard basis for $\mathfrak{g}$. Then

$$x_i(t)s_i = x_{-i}(t^{-1})x_i(-t)t^{\alpha_i^\vee},$$

here $t^{\alpha_i^\vee} = \exp(tH_{\alpha_i})$. There exists a unique j such that $w_0 s_i = s_j w_0$ (since $w_0\Delta = -\Delta$), and $w_0 x_i(t) = x_{-j}(t)w_0$ (up to possibly a coefficient on the t). Therefore

$$\begin{aligned} s_j x_{-j}(t) w_0 s_i &= s_j w_0 x_{-i}(t^{-1}) x_i(-t) t^{\alpha_i^\vee} \\ x_j(t) s_j w_0 s_i &= w_0 s_i x_{-i}(t^{-1}) x_i(-t) t^{\alpha_i^\vee} \\ f(w_0) &= \omega_i(t^{\alpha_i^\vee}) f(w_0 s_i x_{-i}(t^{-1})) \\ t^{-1} f(w_0) &= f(w_0 s_i x_{-i}(t^{-1})); \end{aligned}$$

so the limit as $t \to \infty$ shows that $f(w_0 s_i) = 0$. Since f is N-invariant, f vanishes on the entirety of $Bw_0 s_i B$, so necessarily on the closure $X_{w_0 s_i}$. Note that f vanishes to order 1 on exactly this variety. □

2. Lecture 2

2.1. The Bruhat order.

Definition 2.1. *For* $w_1, w_2 \in W$ *and* $\gamma \in \Phi^+$, *the notation*

$$w_1 \xrightarrow{\gamma} w_2$$

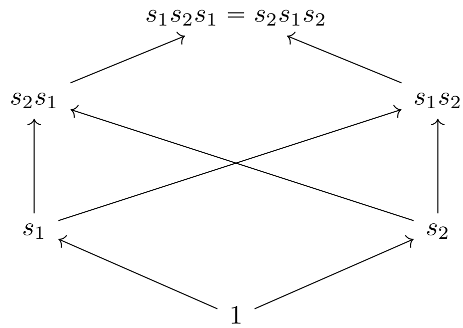
means $s_\gamma w_1 = w_2$ *and* $\ell(w_2) = \ell(w_1) + 1$. *By* $w < w'$ *we mean there is a sequence*

$$w = w_1 \xrightarrow{\gamma_1} w_2 \cdots \xrightarrow{\gamma_k} w_{k+1} = w'.$$

Exercise 2.2. *Show that* $\leqslant$ *establishes a partial order on* W.

Remark 2.3. *We get the same partial order if we stipulate instead that* $w_1 s_\gamma = w_2$ *and* $\ell(w_2) = \ell(w_1) + 1$.

The Hasse diagram for type A_2:



We recall a few results from any standard course in Lie algebras; see for example [Hum72, §9.2, 10.2, 10.3]:

Proposition 2.4.

(1) For any $\gamma \in \Phi^+, w \in W$, $s_{w\gamma} = ws_\gamma w^{-1}$.
(2) For $\alpha_i \in \Delta$, s_i permutes the positive roots other than α_i.
(3) For some (possibly repeated) indices $\{i_j\}$, if

$$s_{i_1} \cdots s_{i_{q-1}} \alpha_{i_q}$$

is negative, then for some index $1 \leqslant p < q$,

$$s_{i_1} \cdots s_{i_q} = s_{i_1} \cdots s_{i_{p-1}} s_{i_{p+1}} \cdots s_{i_{q-1}}.$$

(4) If $w = s_{i_1} \cdots s_{i_q}$ is a minimal-length expression of w, $w(\alpha_{i_q})$ is negative.
(5) $\ell(w)$ equals the size of $\Phi^+ \cap w^{-1}\Phi^-$, and $\ell(w) = \ell(w^{-1})$.

Lemma 2.5. *Let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced decomposition. Set $\gamma_j = s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j}$ ($\gamma_1 = \alpha_{i_1}$). Then the roots $\gamma_1, \ldots, \gamma_\ell$ are distinct and comprise the set $\Phi^+ \cap w\Phi^-$.*

Proof. Distinctness:

Assume $\gamma_j = \gamma_k$, $j < k$. Then we arrive at

$$\alpha_{i_j} = s_{i_j} \cdots s_{i_{k-1}} \alpha_{i_k}$$
$$\alpha_{i_j} = s_{i_{j+1}} \cdots s_{i_k} \alpha_{i_k},$$

which contradicts length-minimality of the subword $s_{i_{j+1}} \cdots s_{i_k}$.

One inclusion:

Each $\gamma_j = s_{i_1} \cdots s_{i_j}(-\alpha_{i_j})$, which is positive by length-minimality. Also,

$$\gamma_j = w(s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_j} \alpha_j),$$

and the part in parentheses is negative again by length-minimality. So $\{\gamma_j\} \subset \Phi^+ \cap w\Phi^-$.

The result follows since each set has size $\ell(w) = \ell$. ☐

Corollary 2.6.

(1) Let w be as before and $\gamma \in \Phi^+$ such that $w^{-1}\gamma \in \Phi^-$. Then for some j,

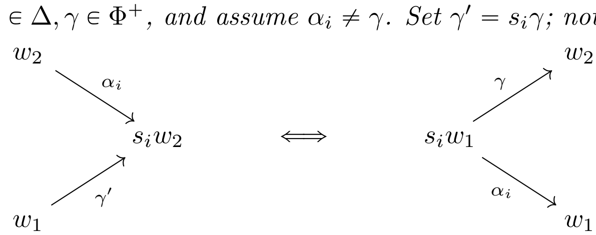
$$s_\gamma s_{i_1} \cdots s_{i_j} = s_{i_1} \cdots s_{i_{j-1}}$$

(2) For $w \in W$ and $\gamma \in \Phi^+$, $\ell(w) < \ell(s_\gamma w)$ if and only if $w^{-1}\gamma \in \Phi^+$.

Exercise 2.7. *Prove the corollary.*

Now we examine neighbourhoods in the Hasse diagram:

Lemma 2.8. *Let $w_1, w_2 \in W, \alpha_i \in \Delta, \gamma \in \Phi^+$, and assume $\alpha_i \neq \gamma$. Set $\gamma' = s_i\gamma$; note $\gamma' \in \Phi^+$. Then*



Proof. Let us prove ($\Longrightarrow$). Note that $\ell(w_2) = \ell(w_1)$ is given. Since $w_2 = s_is_{\gamma'}w_1 = s_\gamma s_i w_1$ it suffices to prove $\ell(s_iw_1) < \ell(w_2)$. Since $s_iw_1 = s_\gamma w_2$, we verify this by checking $w_2^{-1}\gamma$ is negative: $w_2^{-1}\gamma = w_2^{-1}s_i\gamma' \in \Phi^-$. ☐

We take that local statement and extend it a little bit:

Lemma 2.9. *Given $w < w'$ and $\alpha_i \in \Delta$,*

(1) either $s_i w \leqslant w'$ or $s_i w < s_i w'$;
(2) either $w \leqslant s_i w'$ or $s_i w < s_i w'$.

Proof. Let us prove the first statement.
Take a path
$$w = w_1 \to w_2 \to \cdots \to w_k = w'.$$
Of course, if $s_i w \to w$ or if $s_i w = w_2$, the first case holds. So assume $w \to s_i w$ and $s_i w \neq w_2$.
We will induct on k.
Case $k = 2$: apply previous lemma immediately.
Inductive step: we have $s_i w < s_i w_2$. Now consider the pair (w_2, w'). □

Corollary 2.10. *Let $\alpha_i \in \Delta$ and $w_1 \xrightarrow{\alpha_i} w_1'$, $w_2 \xrightarrow{\alpha_i} w_2'$. If one of w_1, w_1' is smaller than one of w_2, w_2', then*
$$w_1 \leqslant w_2 \ \text{ and } w_1' \leqslant w_2'.$$

Proof. Case $w_1' \leqslant w_2$: trivial.
Case $w_1' \leqslant w_2'$: equality gives trivial. Else $w_1' \leqslant w_2$ (previous case) or $w_1 < w_2$, as desired.
Case $w_1 \leqslant w_2'$: we cannot have equality, since then $w_1' = w_2$ but have different lengths. So $w_1 < w_2'$ and either $w_1' \leqslant w_2'$ (previous case) or $w_1' < w_2$ (previous case).
Case $w_1 \leqslant w_2$: then $w_1 < w_2'$ (previous case). □

Finally we come to a third characterization of the Bruhat order:

Proposition 2.11. *Let $w \in W$, and let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word. For ease of notation, set $t_j := s_{i_j}$ (the t_js are not all necessarily distinct).*

(1) If $1 \leqslant j_1 < \ldots < j_k \leqslant \ell$ and $w' = t_{j_1} \cdots t_{j_k}$, then
$$w' \leqslant w$$
(2) If $w' < w$, then w' has a representation of the above form for some $\{j_1, \ldots, j_k\}$.
(3) If $w' \to w$, then there exists a unique index p such that
$$w' = t_1 \cdots t_{p-1} t_{p+1} \cdots t_\ell.$$

Proof. Claim (3) is essentially Corollary 2.6(1). Uniqueness follows from a contradiction given $t_{p+1} \cdots t_q = t_p \cdots t_{q-1}$.
Claim (2) follows from claim (3), since the expression in (3) is already reduced!
Claim (1) we will show by induction (the base case is obvious). If $j_1 > 1$, then $w' \leqslant t_2 \cdots t_\ell$ by hypothesis, and $w' \leqslant t_1 w < w$. If $j_1 = 1$, then $t_1 w' \leqslant t_1 w$ by hypothesis, and $w' \leqslant w$ by Corollary 2.10. □

There is a fourth characterization of the Bruhat order, more geometric in nature.

Proposition 2.12 ([Ste67])**.** *Let $w, w' \in W$. Then*
$$w \leqslant w' \iff X_w \subseteq X_{w'}.$$

BGG reprove this result explicitly ([BGG73, Theorem 2.11]). Another proof follows from the same kind of limit analysis as in our proof of Lemma 1.5.

2.2. **Divided difference operators.** Let $R = \mathbb{Q}[\alpha_i]$, $I = R^W$, $I_+ \subset I$ the subring of elements with no constant term, and J the ideal generated by I_+.

Definition 2.13. *For $\gamma \in \Phi$, we define for $f \in R$*
$$A_\gamma f = \frac{f - s_\gamma f}{\gamma}$$
(note that this is well-defined!).

Lemma 2.14.

(1) $A_{-\gamma} = -A_\gamma$, $A_\gamma^2 = 0$.
(2) $w A_\gamma w^{-1} = A_{w\gamma}$.
(3) $s_\gamma A_\gamma = -A_\gamma s_\gamma = A_\gamma$; $s_\gamma = -\gamma A_\gamma + 1 = A_\gamma \gamma - 1$.
(4) $A_\gamma f = 0 \iff s_\gamma f = f$.
(5) $A_\gamma J \subseteq J$.

(6) For $\chi \in \mathfrak{h}_{\mathbb{Q}}^*$, the commutator of A_γ with multiplication by χ is

$$[A_\gamma, \chi] = \chi(\gamma^\vee)s_\gamma.$$

Exercise 2.15. *Prove the lemma.*

3. Lecture 3

3.1. **Divided difference operators, continued.** For simplicity, A_i means A_{α_i} for simple root $\alpha_i \in \Delta$.

The divided difference operators satisfy the following crucial property:

Theorem 3.1. *Let $\alpha_{i_1}, \dots, \alpha_{i_\ell}$ be a sequence of simple roots. Set $w = s_{i_1} \cdots s_{i_\ell}$ and $A_{(i_1,\dots,i_\ell)} := A_{i_1} \circ \cdots \circ A_{i_\ell}$. Then*

(1) If $\ell(w) < \ell$, $A_{(i_1,\dots,i_\ell)} = 0$.

(2) If $\ell(w) = \ell$, $A_{(i_1,\dots,i_\ell)}$ does not depend on the decomposition $w = s_{i_1} \cdots s_{i_\ell}$; in this case we define $A_w := A_{(i_1,\dots,i_\ell)}$.

Proof. We proceed by induction on ℓ, the case $\ell = 1$ being trivial.

For (1), examine $v = s_{i_1} \cdots s_{i_{\ell-1}}$. If $\ell(v) < \ell - 1$, then by induction we are done. Else, $\ell(v) = \ell - 1$ and $\ell(w) = \ell - 2$. Since $\ell(w^{-1}) < \ell(s_{i_\ell} w^{-1})$, $w\alpha_{i_\ell} > 0$. So for some index j,

$$s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_j} = s_{i_{\ell-1}} \cdots s_{i_{j+1}},$$

thus $s_{i_j} \cdots s_{i_{\ell-1}} = s_{i_{j+1}} \cdots s_{i_\ell}$; furthermore, these are both reduced decompositions. By induction,

$$A_{i_j} \circ \cdots \circ A_{i_{\ell-1}} = A_{i_{j+1}} \circ \cdots \circ A_{i_\ell};$$

therefore $A_{i_j} \circ \cdots \circ A_{i_{\ell-1}} = A_{i_{j+1}} \circ \cdots \circ A_{i_\ell}^2 = 0$.

For (2), we introduce operators

$$B_{(i_1,\dots,i_\ell)} := s_{i_\ell} \cdots s_{i_1} A_{(i_1,\dots,i_\ell)}.$$

Put $w_j := s_{i_\ell} \cdots s_{i_j}$. Then

$$\begin{aligned} B_{(i_1,\dots,i_\ell)} &= w_2 A_{i_1} w_2^{-1} w_3 A_{i_2} w_3^{-1} \cdots w_\ell A_{i_{\ell-1}} w_\ell^{-1} A_{i_\ell} \\ &= A_{i_1}^{w_2} \circ A_{i_2}^{w_3} \circ \cdots \circ A_{i_{\ell-1}}^{w_\ell} \circ A_{i_\ell}, \end{aligned}$$

where $A_\gamma^w := wA_\gamma w^{-1}$.

Lemma 3.2.

$$\left[B_{(i_1,\cdots i_\ell)}, \chi\right] = \sum_{j=1}^{\ell} \chi(w_{j+1}\alpha_j^\vee) w_{j+1} w_j^{-1} B_{(i_1,\cdots,\hat{i_j},\cdots,i_\ell)}$$

Proof. First,

$$\begin{aligned} \left[B_{(i_1,\cdots i_\ell)}, \chi\right] &= \left[A_{i_1}^{w_2} \circ \cdots \circ A_{i_\ell}, \chi\right] \\ &= \sum_{j=1}^{\ell} A_{i_1}^{w_2} \cdots [A_{i_j}^{w_{j+1}}, \chi] \cdots A_{i_\ell} \\ &=: \sum_{j=1}^{\ell} T_j. \end{aligned}$$

Observe that $s_{w_{j+1}\alpha_j} = w_{j+1} w_j^{-1}$ and $[A_{i_j}^{w_{j+1}}, \chi] = \chi(w_{j+1}\alpha_j^\vee) s_{w_{j+1}\alpha_j}$. Therefore

$$T_j = \chi(w_{j+1}\alpha_j^\vee) A_{i_1}^{w_2} \cdots A_{i_{j-1}}^{w_j} w_{j+1} w_j^{-1} A_{i_{j+1}}^{w_{j+2}} \cdots A_{i_\ell}.$$

One checks that

$$A_{i_k}^{w_{k+1}} w_{j+1} w_j^{-1} = w_{j+1} w_j^{-1} A_{i_k}^{s_{i_\ell} \cdots \widehat{s_{i_j}} \cdots s_{i_{k+1}}},$$

so we may more simply write

$$T_j = \chi(w_{j+1}\alpha_j^\vee) w_{j+1} w_j^{-1} B_{(i_1,\cdots,\hat{i_j},\cdots,i_\ell)},$$

as desired.

□

Now fix a $1 \leqslant j \leqslant \ell$. If $\ell(s_{i_1} \cdots \widehat{s_{i_j}} \cdots s_{i_\ell}) < \ell - 1$, then $T_j = 0$ by inductive hypothesis.

Otherwise, $s_{i_1} \cdots \widehat{s_{i_j}} \cdots s_{i_\ell} = w' \xrightarrow{\gamma} w$ where

$$\gamma = s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j},$$

and

$$\chi(w_{j+1}\alpha_j^\vee) = w'\chi(w'w_{j+1}\alpha_j^\vee) = w'\chi(s_{i_1} \cdots s_{i_{j-1}}\alpha_j^\vee) = w'\chi(\gamma^\vee).$$

Furthermore,

$$w_{j+1}w_j^{-1}B_{(i_1,\cdots,\hat{i_j},\cdots,i_\ell)} = w_{j+1}w_j^{-1}w'^{-1}A_{(i_1,\cdots,\hat{i_j},\cdots,i_\ell)} = w^{-1}A_{w'}$$

using the inductive hypothesis.

Since all $w' \xrightarrow{\gamma} w$ appear in the form above, we have

$$\left[B_{(i_1,\cdots i_\ell)}, \chi\right] = \sum_{w' \xrightarrow{\gamma} w} w'\chi(\gamma^\vee)w^{-1}A_{w'}.$$

The RHS does not depend on the choice of reduced word for w! The result follows then from the following lemma. □

Lemma 3.3. *Let B be an operator on R such that $B(1) = 0$ and $[B, \chi] = 0$ for all χ. Then $B = 0$.*

Proof. Thus B vanishes on any monomial $\chi_1 \cdots \chi_s$. □

We actually derived the following useful corollary, which we will use later to produce a formula for multiplication by Chern classes.

Corollary 3.4.

$$\left[w^{-1}A_w, \chi\right] = \sum_{w' \xrightarrow{\gamma} w} w'\chi(\gamma^\vee)w^{-1}A_{w'}$$

3.2. **Schubert classes in homology as operators on** R. Let $S_i = R_i^*$, where R_i is the $(2)i^{\text{th}}$ graded piece of R; set $S = \oplus S_i$. There is a natural pairing

$$(,) : S_i \times R_i \to \mathbb{Q}$$

which we extend to $S \times R \to \mathbb{Q}$ by 0. Since $H_{2i}(X)$ and $H^{2i}(X)$ have a perfect pairing, we expect to see a dual basis in S to the cohomology Schubert classes in R (more precisely in R/J). One can show that the dual basis to $\{[X_w]\}$ under this pairing is $\{\mu(X_{w_0w})\}$; this follows from some relatively straightforward intersection theory, see for example [Ful98, Chapter 19]. In fact, the perspective of [BGG73] is to produce the basis $\{\mu(X_w)\}$ in S and use it to make explicit the dual polynomials $\{[X_{w_0w}]\}$ in R/J.

Toward that end, we define some operators on S. First of all, W acts naturally on S since it does on R. For $\chi \in \mathfrak{h}_{\mathbb{Q}}^*$, we denote by

$$\chi^* : S \to S$$

the adjoint to the operator $\chi\cdot : R \to R$. Likewise, we let F_γ be the adjoint of A_γ, and F_w that of A_w. We already have the following results:

Theorem 3.5. *Let $\alpha_{i_1}, \ldots, \alpha_{i_\ell}$ be a sequence of indices, and set $w = s_{i_1} \cdots s_{i_\ell}$.*

(1) If $\ell(w) < \ell$, $F_{i_\ell} \cdots F_{i_1} = 0$.
(2) If $\ell(w) = \ell$, $F_{i_\ell} \cdots F_{i_1}$ depends only on w and equals F_w.
(3) $[\chi^*, F_w w] = \sum_{w' \xrightarrow{\gamma} w} w'\chi(\gamma^\vee)F_{w'}w$

Set $D_w := F_w(\mathbb{1})$. We will later see that D_w is identified with the Schubert fundamental classes $\mu(X_w)$. We record various properties of the D_w as follows:

Theorem 3.6.

(1) $D_w \in S_{\ell(w)}$.
(2) Let $w \in W$ and $\alpha \in \Delta$. Then

$$F_\alpha D_w = \begin{cases} 0, & \ell(ws_\alpha) = \ell(w) - 1 \\ D_{ws_\alpha}, & \ell(ws_\alpha) = \ell(w) + 1. \end{cases}$$

(3) For $\chi \in \mathfrak{h}_{\mathbb{Q}}^*$, $\chi^* D_w = \sum_{w' \xrightarrow{\gamma} w} w'\chi(\gamma^\vee)D_{w'}$
(4) For $w \in W$ and $\alpha \in \Delta$,

$$s_\alpha D_w = \begin{cases} -D_w, & \ell(ws_\alpha) = \ell(w) - 1 \\ -D_w + \sum_{w' \xrightarrow{\gamma} ws_\alpha} w'\alpha(\gamma^\vee)D_{w'}, & \ell(ws_\alpha) = \ell(w) + 1. \end{cases}$$

(5) *Let* $w \in W$, $\ell = \ell(w)$, *and choose* $\chi_1, \dots, \chi_\ell \in \mathfrak{h}^*_{\mathbb{Q}}$. *Then*

$$(D_w, \chi_1 \cdots \chi_\ell) = \sum \chi_1(\gamma_1^\vee) \cdots \chi_\ell(\gamma_\ell^\vee),$$

the sum running over all chains $e \xrightarrow{\gamma_1} w_1 \cdots \xrightarrow{\gamma_\ell} w_\ell = w^{-1}$.

Proof. (1) is obvious.

(2) follows from Theorem 3.5(1).

(3) follows from Theorem 3.5(3) since $\chi^*(\mathbb{1}) = 0$.

For (4), Lemma 2.14(3) shows that the action of s_α on S is equal to that of $\alpha^* F_\alpha - \mathrm{id}$. So the result follows from (2) and (3) combined.

Finally, (5). Observe that if $w' \xrightarrow{\gamma} w$, then $w' s_{w'^{-1}\gamma} = w$, therefore

$$w'^{-1} \xrightarrow{w'^{-1}\gamma} w^{-1},$$

and (note $w'^{-1}\gamma$ is positive)

$$\chi(w'^{-1}\gamma^\vee) = w'\chi(\gamma^\vee),$$

so the result follows from induction on ℓ and (3).

□

Define a new set of operators $\widehat{D}_w$ on R by the rule

$$\widehat{D}_w(f) = \theta(f) \cap \mu(X_w)$$

if f is homogeneous of degree $\ell(w)$, extending by 0 in the natural way. Thus we may think of $\widehat{D}_w \in S_{\ell(w)}$. The following is the geometric crux of the entire paper:

Theorem 3.7. $\widehat{D}_w = D_w$.

Proof. The trick is that it suffices to show $\widehat{D}_w = \mathbb{1}$ and

$$\chi^* \widehat{D}_w = \sum_{w' \xrightarrow{\gamma} w} w'\chi(\gamma^\vee)\widehat{D}_{w'}$$

for every $\chi \in \mathfrak{h}^*_{\mathbb{Z}}$, thanks to induction on $\ell(w)$ and Theorem 3.6(3).

Recall from algebraic topology that if $z \in H^k(X)$, $c \in H^i(X)$, and $y \in H_{k+i}(X)$,

$$z\,(c \cap y) = (c \cup z)(y);$$

therefore for any $z = \theta(f) \in H^{2\ell(w)-2}(X)$,

$$(\chi^* \widehat{D}_w, f) = (\widehat{D}_w, \chi \cdot f) = (c_1(\mathcal{L}_\chi) \cup z) \cap \mu(X_w) = z \cup (c_1(\mathcal{L}_\chi) \cap \mu(X_w)),$$

and the proof reduces to showing

$$\mu(X_w) \cap c_1(\mathcal{L}_\chi) = \sum_{w' \xrightarrow{\gamma} w} w'\chi(\gamma^\vee)\mu(X_{w'}). \tag{1}$$

By functoriality (i.e., $f_*(f^*\delta \cap \epsilon) = \delta \cap f_*\epsilon$ whenever this equations makes sense), we may interpret (1) as taking place in the homology and cohomology of X_w itself. A result from algebraic topology says that if σ is a nonzero section of $\mathcal{L}_\chi$ and it has divisor

$$\mathrm{div}\,\sigma = \sum m_i Y_i,$$

then

$$X_w \cap c_1(\mathcal{L}_\chi) = \sum m_i \mu(Y_i),$$

provided that X_w is nonsingular in codimension 1. For a proof of this latter fact, see [BGG73, Proposition 4.3].

Now, both sides of (1) are linear in χ, so writing $\chi = \lambda - \lambda'$ for regular dominant λ, λ', we may reduce to the case $\chi = \lambda$ is regular dominant. Let V be an irreducible representation of G with highest weight λ, and consider the line bundle η_V on $\mathbb{P}(V)$ given by the total space

$$\eta_V = \{(P, \phi) | \phi : P \to \mathbb{C}, P \subseteq V, \dim P = 1\};$$

that is, $\eta_v = \mathcal{O}(1)$. Since B acts trivially on the line $\langle v_\lambda \rangle$ of highest weight in V, there is a natural embedding $i : G/B \hookrightarrow \mathbb{P}(V)$. It is easy to verify that $i^*(\eta_V) = E_\lambda$.

Define a linear functional $\phi_w : V \to \mathbb{C}$ by

$$\phi_w(v_\mu) = \begin{cases} 1 & \mu = w\lambda \\ 0 & v_\mu \text{ is a weight vector with weight different than } w\lambda, \end{cases}$$

extending by linearity. Then ϕ_w induces a global section σ of $\mathcal{O}(1)$. By definition, σ vanishes nowhere on the cell $NwB/B \subseteq X_w$. Therefore $\operatorname{div}(\sigma)$ is supported in

$$\bigcup_{w' \xrightarrow{\gamma} w} X_{w'};$$

furthermore, the $X_{w'}$s are irreducible, so $\operatorname{div}(\sigma) = \sum_{w' \xrightarrow{\gamma} w} a_\gamma X_{w'}$ for some integers $a_\gamma \geqslant 0$. We can calculate a_γ as the multiplicity of vanishing of the section $\delta^*\sigma$ pulled back via the map $\delta : \mathbb{P}^1 \to X_w$ that sends 0 to $w'B/B$. This in turn equals the order of vanishing of

$$\phi_w(\exp(tE_{-\gamma})v_{w'\lambda}) = ct^{w'\lambda(\gamma^\vee)}$$

at $t = 0$, so the proof is complete.

□

3.3. **Polynomials in cohomology.** Let $\mathcal{H}$ denote the subspace of S dual to J under the pairing $(,)$. Since $\mathbb{1}$ is clearly in $\mathcal{H}$ and since the F_γ fix $\mathcal{H}$, we see that each $D_w \in \mathcal{H}$.

Theorem 3.8. *The set $\{D_w\}$ forms a basis of $\mathcal{H}$.*

Proof. We first argue that the D_w are linearly independent (I suppose this already follows from having identified them with the Schubert fundamental classes, but here is a direct algebraic proof). By Theorem 3.6(5), $(D_{w_0}, \rho^{\ell(w_0)}) > 0$ (the pairing of ρ with any positive root is strictly positive). So in particular $D_{w_0} \neq 0$.

Assume for contradiction that there is a nontrivial dependence relation

$$\sum c_w D_w = 0,$$

and find a $\tilde{w}$ of minimal length so that $c_{\tilde{w}} \neq 0$; set $\ell = \ell(\tilde{w})$ and $k = \ell(w_0)$. Find simple reflections $s_{i_1}, \ldots s_{i_{k-\ell}}$ so that

$$\tilde{w} s_{i_1} \cdots s_{i_{k-\ell}} = w_0.$$

Then by repeated application of Theorem 3.5,

$$F_{i_{k-\ell}} \cdots F_{i_1} D_{\tilde{w}} = D_{w_0};$$
$$F_{i_{k-\ell}} \cdots F_{i_1} D_w = 0$$

for any $w \neq \tilde{w}$ such that $\ell(w) \geqslant \ell$. Applying $F_{i_{k-\ell}} \cdots F_{i_1}$ to our sum, therefore, we arrive at $c_{\tilde{w}} D_{w_0} = 0$, a contradiction.

Second, we show that the D_w span. To verify that

$$\bigoplus \mathbb{Q} D_w \to \mathcal{H}$$

is surjective, it suffices to show that the dual map is injective. That is, we must show that if $f \in R$ satisfies $(D_w, f) = 0$ for all w, then $f \in J$. It suffices to prove this just for homogeneous f, and we do so by induction on degree of f, the case $\deg f = 0$ being clear.

Suppose f has degree $d > 0$. For any $\alpha \in \Delta$ and $w \in W$, we observe that

$$(D_w, A_\alpha f) = \begin{cases} (D_{ws_\alpha}, f) = 0 & \ell(ws_\alpha) = \ell(w) + 1 \\ 0 & \ell(ws_\alpha) = \ell(w) - 1; \end{cases}$$

therefore by induction $A_\alpha f \in J$. Rearranging, $f - s_\alpha f \in J$. It follows readily that $f - wf \in J$ for any $w \in W$; now we average:

$$f - \frac{1}{|W|} \sum_{w \in W} wf \in J,$$

but of course the term $\frac{1}{|W|} \sum_{w \in W} wf$ is itself in J. So $f \in J$.

□

By construction, $\mathcal{H}$ and R/J have a nondegenerate pairing induced from $(,)$. *Already, this gives a new proof that R/J has finite $\mathbb{Q}$-dimension.* Now let $P_w \in R/J$ be the dual basis to D_w. We immediately know the following:

Theorem 3.9.

(1)

$$A_\alpha P_w = \begin{cases} 0, & \ell(ws_\alpha) = \ell(w) + 1 \\ P_{ws_\alpha}, & \ell(ws_\alpha) = \ell(w) - 1 \end{cases}$$

(2)

$$\chi P_w = \sum_{w \xrightarrow{\gamma} w'} w\chi(\gamma^\vee) P_{w'}$$

(3)

$$s_\alpha P_w = \begin{cases} P_w, & \ell(ws_\alpha) = \ell(w)+1 \\ P_w - \sum_{ws_\alpha \xrightarrow{\gamma} w'} w\alpha(\gamma^\vee) P_{w'}, & \ell(ws_\alpha) = \ell(w)-1 \end{cases}$$

4. Lecture 4

4.1. **Structure of R.** Amazingly, then, we have the following corollary:

Corollary 4.1.

$$P_w = A_{i_\ell} \cdots A_{i_1} P_{w_0},$$

where $w = w_0 s_{i_1} \cdots s_{i_\ell}$ and $\ell(w) = \ell(w_0) - \ell$. In other words, $P_w = A_{w^{-1}w_0} P_{w_0}$. More generally, $P_v = A_{v^{-1}w} P_w$ when v is a left subword of some reduced expression for w.

So in our quest for *explicit* polynomials, we reduce to finding P_{w_0}.

Theorem 4.2.

$$P_{w_0} \equiv \frac{1}{|W|} \prod_{\gamma \in \Phi^+} \gamma \mod J.$$

The proof is somewhat lengthy and technical, see [BGG73, Theorem 3.15].

Some other properties of the polynomials are:

Proposition 4.3.

1. *Set $k = \ell(w_0)$. Another expression for P_{w_0} is*

$$P_{w_0} \equiv \rho^k / k! \mod J$$

2. *Suppose $w_1, w_2 \in W$, $\ell(w_1) + \ell(w_2) = k$. Then $P_{w_1} P_{w_2} = 0$ if $w_1 \neq w_0 w_2$, and $P_{w_1} P_{w_0 w_1} = P_{w_0}$.*
3. *For any $f \in R$, one may write*

$$f = \sum_{w \in W} \tilde{P}_w f_w$$

where $f_w \in R^W$ and the $\tilde{P}_w$ are fixed homogeneous lifts of the P_w.

Proof. Let us prove (3), for example.

We proceed by induction on $\deg f$. If f has degree 1, the statement is clear.

Take $\deg f = k$, and let g collect the deg $< k$ terms of f. Then $f - g$ is homogeneous of degree k. Then

$$(D_w, f - g) = 0$$

unless $\ell(w) = k$, so take $w_1, \ldots, w_t$ to be the elements in W of length k, and set $c_i = (D_{w_i}, f - g)$. Define

$$p = f - g - \sum c_i \tilde{P}_{w_i};$$

then clearly $(D_w, p) = 0$ for all $w \in W$. Note that p is still homogeneous of degree k. We can therefore write

$$p = \sum r_j s_j$$

with each $r_j \in R$, $s_j \in R^W$, s_j homogeneous, and $s_j(0) = 0$. Then each r_j can be chosen to have degree $< k$, so we may write

$$r_j = \sum \tilde{P}_w r_{j,w}$$

for each j. Then

$$f = g + \sum c_i \tilde{P}_{w_i} + \sum \tilde{P}_w \sum r_{j,w} s_j$$

puts f in the desired form. □

Now suppose we have picked specific lifts $\tilde{P}_{w_0} = \frac{1}{|W|} \prod_{\Phi^+} \gamma$ and in general $\tilde{P}_w := A_{w^{-1}w_0} P_{w_0}$. The following theorem shows how to decompose R as a free R^W module of rank $|W|$:

Theorem 4.4. *The multiplication map*

$$\begin{aligned} R^W \oplus \cdots \oplus R^W &\to R \\ (f_{w_0}, \ldots, f_e) &\mapsto \sum f_w \tilde{P}_w \end{aligned}$$

is an isomorphism of R^W-modules.

Proof. Surjectivity follows from the previous proposition.

For injectivity, we simply describe the inverse map by algorithm. Take $f \in R$. We know $f = \sum f_w \tilde{P}_w$ for *some* choices of $f_w \in R^W$; we show how to recover them from f (so they are uniquely determined).

First, $f_{w_0} = A_{w_0}$. To find $f_{w_0 s_i}$ for the various s_i, first set $f' = f - f_{w_0}$. Then $f_{w_0 s_i} = A_{w_0 s_i} f'$. In general, $A_w f = f_w$ if $f_v = 0$ for all $\ell(v) > \ell(w)$: if w and w' have the same length, then $A_w P_{w'} = A_w A_{w'^{-1} w_0} P_{w_0}$, but $ww'^{-1} w_0 \neq w_0$ means the operator $A_w A_{w'^{-1} w_0}$ is 0. □

Corollary 4.5. *By the algorithm, if f is homogeneous of degree k, then the (nonzero) f_w are each homogeneous of degree $k - \ell(w)$.*

Example 4.6. *Applying the above algorithm to x^3 in the ring $R = \mathbb{Q}[x, y, z]$, $W = S_3$, we have*

$$x^3 = (x + y + z)x^2 + (-yx - zx - zy)x + (xyz)1$$

This incidentally shows that $x^3 \in J$.

Exercise 4.7. *Express x^4, y^2, in the basis of the $\tilde{P}_w$.*

4.2. **Integration formula.** Take $f \in R$ and express $f = \sum_{w \in W} \tilde{P}_w f_w$ as above. Then $\int_X f = f \cap \mu(X_{w_0}) = A_{w_0} f = f_{w_0}(0)$. There is another way to calculate $f_{w_0}(0)$, (i.e., to apply D_{w_0}):

Theorem 4.8.

$$\frac{1}{\prod_{\Phi^+} \gamma} \sum_{w \in W} (-1)^{\ell(w)} w f \Big|_0 = f_{w_0}(0)$$

Proof. We will first prove that

$$\frac{1}{\prod_{\Phi^+} \gamma} \sum_{w \in W} (-1)^{\ell(w)} w f = f_{w_0}.$$

Fix a $w \neq w_0$; we can find some A_i such that $A_i P_w = 0$. Fix a subset $\tilde{W} \subset W$ such that every $u \in W$ is either in $\tilde{W}$ or in $\tilde{W} s_i$, but not both; i.e., $W = \tilde{W} \sqcup \tilde{W} s_i$. Then

$$\begin{aligned} \frac{1}{\prod_{\Phi^+} \gamma} \sum_{u \in W} (-1)^{\ell(u)} u P_w &= \frac{1}{\prod_{\Phi^+} \gamma} \sum_{u \in \tilde{W}} (-1)^{\ell(u)} \left[u P_w - u s_i P_w \right] \\ &= \frac{1}{\prod_{\Phi^+} \gamma} \sum_{u \in \tilde{W}} (-1)^{\ell(u)} u \alpha_i A_i P_w \\ &= 0. \end{aligned}$$

Therefore the result follows. □

Remark 4.9. *Evaluation at a point other than 0, given* $\deg f \leqslant k$*, is valid.*

Remark 4.10. *This formula can also be distilled from the localization theorem in T-equivariant cohomology, after passing back to regular cohomology.*

Corollary 4.11. *Suppose $g \in J$. Then $g_w(0) = 0$ for all $w \in W$. Therefore:*

(1) The alternating sum

$$\frac{1}{|W|} \sum_{w \in W} (-1)^{\ell(w)} w g \Big|_0 = 0$$

(2) If $\deg g \leqslant \ell(w_0)$*, then*

$$\frac{1}{|W|} \sum_{w \in W} (-1)^{\ell(w)} w g = 0$$

with no evaluation.

Example 4.12. *Back to our $R = \mathbb{Q}[x, y, z]$ and $W = S_3$, $x^2 y + y^2 x$ is in J since*

$$x^2 y + y^2 x - (x^2 y + y^2 x) - (x^2 z + z^2 x) + (y^2 z + z^2 y) + (z^2 x + x^2 z) - (y^2 z + z^2 y) = 0$$

Exercise 4.13. *Suppose f satisfies $s_i f = f$ for some i. Then show $\int f = 0$ two different ways: (1) by applying A_{w_0} and (2) by applying the formula above.*

4.3. **Application to testing the Saturation Conjecture.** Let $\mathcal{C}$ be the semigroup of triples (λ, μ, ν) of dominant weights such that $\lambda + \mu + \nu$ is in the root lattice and

$$c_{N\lambda, N\mu, N\nu} := \dim \left[V(N\lambda) \otimes V(N\mu) \otimes V(N\nu)\right]^G \neq 0$$

for some N. Then by extensive work over the past few decades (see, for example, the survey by Kumar [Kum14]), we know

Theorem 4.14. *$(\lambda, \mu, \nu) \in \mathcal{C}$ if and only if for certain (u, v, w) satisfying*

$$[X_u] \cdot [X_v] \cdot [X_w] = [X_e] \in H^*(G/P_i),$$

the inequality

$$(u^{-1}\lambda + v^{-1}\mu + w^{-1}\nu)(x_i) \leqslant 0$$

holds. Here P_i is the maximal parabolic for simple index i, and x_i the associated fundamental coweight.

Therefore finding the inequalities for $\mathcal{C}$ amounts to performing lots of cup products in cohomology and keeping track of certain ones.

A famous conjecture asks:

Conjecture 4.15. *If G is simply-laced, then*

$$c_{N\lambda, N\mu, N\nu} \neq 0 \implies c_{\lambda, \mu, \nu} \neq 0.$$

This is known to fail for all non-simply-laced types. It had been verified for G of type A (any rank) by Knutson and Tao [KT99] and type D_4 by Kapovich, Kumar, and Millson [KKM09]. I was able to verify it for types D_5 and D_6 by first writing down the inequalities (by doing lots of cup products) and then checking the conjecture on the minimal generating set of $\mathcal{C}$: the Hilbert basis. Details can be found in [Kie19].

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