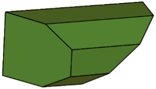
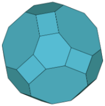
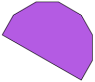
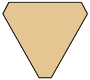
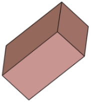
  

# Weight Polytopes of Demazure Modules

**Joshua Kiers**
w/ Marc Besson, Sam Jeralds, and Jiuzu Hong





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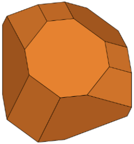

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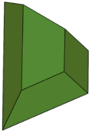
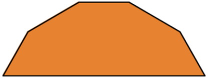
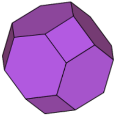


# What is a Demazure module?

Let $G$ be a complex, reductive Lie group , e.g. $SL_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$.
Fix a maximal torus $T \subset G$ , e.g. diagonal matrices.
Fix a Borel subgroup $B \supset T$ , e.g. upper-triangular matrices.
For each weight $\lambda : T \to \mathbb{C}^*$ which is dominant w.r.t. $B$, there is a unique (up-to-isom.) irreducible representation $V_\lambda$ of $G$.

# What is a Demazure module?

Each $V_\lambda$ is a "highest weight" module: there exists a unique (up-to-scaling) vector $v_\lambda \in V_\lambda$ such that

- $b.v_\lambda \in \mathbb{C}v_\lambda$ for all $b \in B$
- $t.v_\lambda = \lambda(t)v_\lambda$ for all $t \in T$

# What is a Demzure module?

Let $W$ be the Weyl group: $W = N_G(T)/T$.
$W$ acts on the lattice of all $T$-weights $X^*(T)$.
Fix $w \in W$.
Note that $w.v_\lambda$ is well-defined up to scaling (and has $T$-weight $w\lambda$).

**Definition**

The Demazure module $V_\lambda^w$ is the smallest $B$-subrepresentation of $V_\lambda$ that contains $w.v_\lambda$.

In other words, $\mathcal{U}(\mathfrak{b})w.v_\lambda$.

# Geometrically speaking...

$V_\lambda$ can be realized as the dual space to $H^0(G/B, \mathcal{L}_\lambda)$, where $G/B$ is the (generalized) flag manifold and $\mathcal{L}_\lambda$ is the line bundle constructed topologically with total space

$$\begin{array}{c} \mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda} \\ \downarrow \\ G/B \end{array}$$

Likewise, $V_\lambda^w$ is dual to

$$H^0(X_w, \mathcal{L}_\lambda\big|_{X_w}),$$

where $X_w$ is the Schubert variety $\overline{BwB/B} \subseteq G/B$.

# Main question

$V_\lambda^w$ is diagonalizable with respect to the $T$-action:

$$V_\lambda^w = \oplus_\mu V_\lambda^w(\mu),$$

where $T$ acts on $V_\lambda^w(\mu)$ via the character $\mu$.

## Question

What is $\dim V_\lambda^w(\mu)$ as a function of $\lambda, w, \mu$?
For which $\mu$ is $\dim V_\lambda^w(\mu) \neq 0$?

Define $\mathrm{char}(V_\lambda^w) = \sum_\mu \dim V_\lambda^w(\mu) e^\mu$, an element of the ring of formal characters $\mathbb{Z}[e^\mu]$.

# Main question

Define $\mathrm{char}(V_\lambda^w) = \sum_\mu \dim V_\lambda^w(\mu) e^\mu$.

**Theorem (Demazure Character Formula)**

*Write $w$ as a reduced word $w = s_{i_1} \cdots s_{i_k}$.*
*Then*

$$\mathrm{char}(V_\lambda^w) = D_{i_1} D_{i_2} \cdots D_{i_k}(e^\lambda)$$

where

$$D_i(f) := \frac{f - e^{-\alpha_i} s_i f}{1 - e^{-\alpha_i}}$$

So calculation of any particular $\mathrm{char}(V_\lambda^w)$ is readily available.

# Main question

This still doesn't answer our two questions combinatorially or efficiently.
In our work we focus on the second question.

## Question

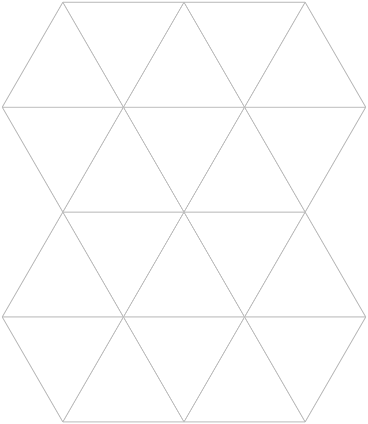
For which $\mu$ is $\dim V_\lambda^w(\mu) \neq 0$?

We know for sure that every $\mu = v\lambda$ will appear with $v \leq w$.
Consider $P_\lambda^w = \mathrm{Conv}\{v\lambda : v \leq w\}$, a convex polytope. Is it possible that

$$V_\lambda^w(\mu) \neq 0 \iff \mu \in P_\lambda^w?$$

# Example

Let $G = GL_3$.
Let $\lambda = \rho = \omega_1 + \omega_2 = (2, 1, 0)$.



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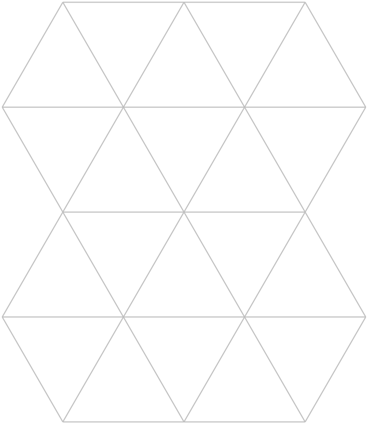

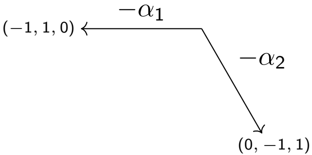

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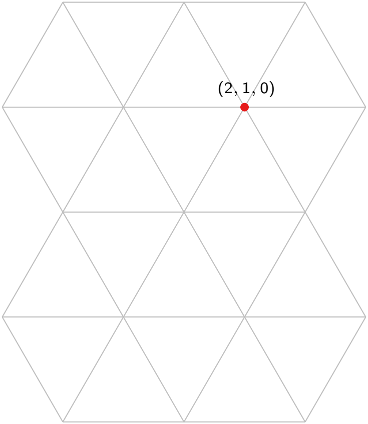


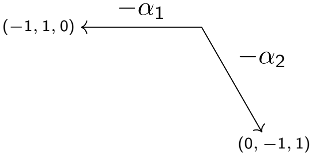


$x_1^2 x_2$ $\quad$ $e$

# Example

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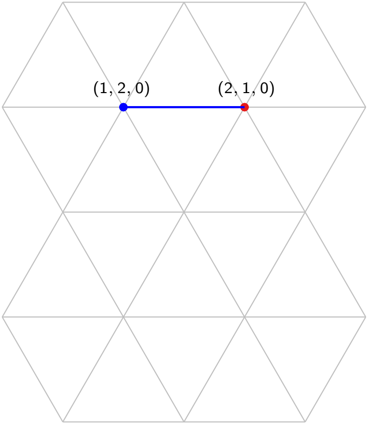


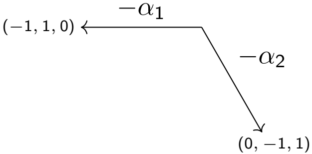


| | |
|---|---|
| $x_1^2 x_2$ | $e$ |
| $x_1^2 x_2 + x_1 x_2^2$ | $s_1$ |

# Example

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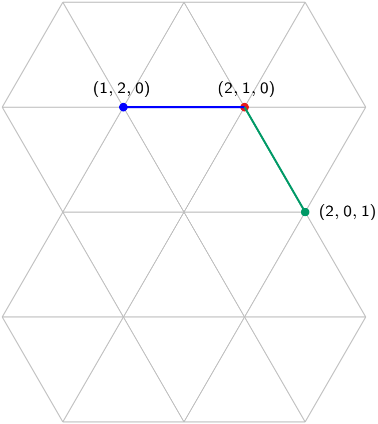


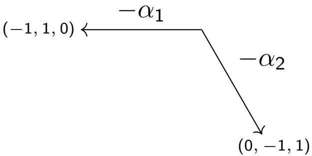


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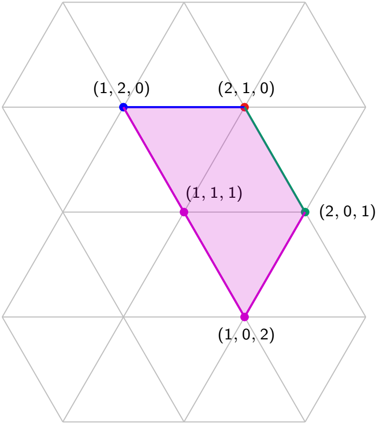


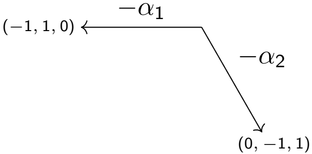


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| $x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_3^2$ | $s_2s_1$ |

# Example

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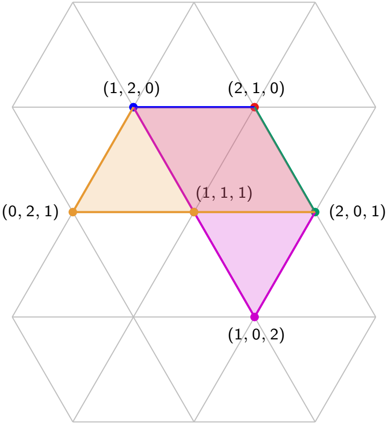


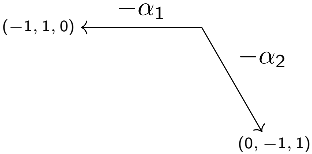


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# Example

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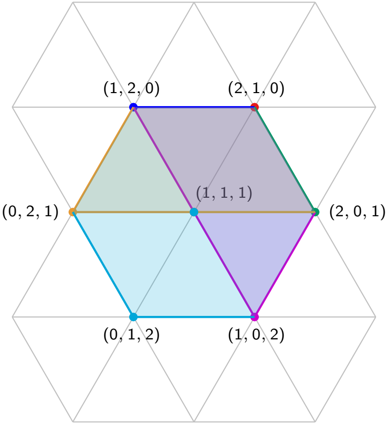


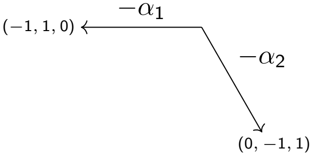


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| $x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2$ | $s_2 s_1$ |
| $x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3$ | $s_1 s_2$ |
| $x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3$ $+ 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$ | $w_0$ |

# Example

Let $G = GL_3$.
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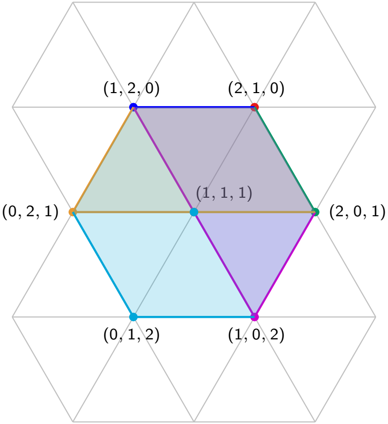


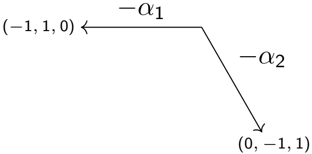


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| $x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_3^2$ | $s_2s_1$ |
| $x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3$ | $s_1s_2$ |
| $x_1^2x_2 + x_1x_2^2 + x_1^2x_3$ $+2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$ | $w_0$ |

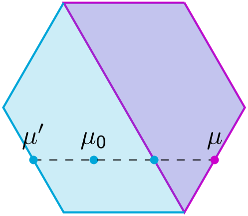
**They are all saturated**

# A Saturation Conjecture

Is the support of $\mathrm{char}(V_\lambda^w)$ as saturated as it could be?
In type A ($G = GL_n$), this was conjectured by C. Monical, N. Tokcan, and A. Yong (2017), with the verbiage of *key polynomials* and their *Newton polytopes*.
It was first proven by A. Fink, C. Mészáros, and A. St. Dizier (2017).
In all types, this is known for $w = w_0$ (the whole $V_\lambda$) (folklore? Rado?).

# Half-space description of $P_\lambda^w$

Perhaps a different description of $P_\lambda^w$ will help prove the conjecture.
**Motivation:**
Say $s_i w \leq w$, and we already know that $\mathrm{char}(V_\lambda^{s_i w})$ is saturated.




$$\begin{array}{c} \mathrm{char}(V_\lambda^{s_i w}) \\ \big\downarrow D_i \\ \mathrm{char}(V_\lambda^{w}) \end{array}$$

Observe that $D_i$ will "spread out" the $\alpha_i$ root strings through weights supported in $\mathrm{char}(V_\lambda^{s_i w})$.
So for each $\mu_0 \in P_\lambda^w$, it suffices to produce $\mu = \mu_0 + k\alpha_i$, where $k \in \mathbb{Z}_{\geq 0}$ and $\mu \in P_\lambda^{s_i w}$. This can be controlled using the inequalities (i.e. faces) that determine $P_\lambda^{s_i w}$.

# Inequalities from GIT

Via the Borel-Weil description,

$$\begin{aligned}\mu \in P_\lambda^w &\iff V_{n\lambda}^w(n\mu) \neq 0 && \text{for some } n \geq 1 \\ &\iff H^0(X_w, \mathcal{L}_\lambda^{\otimes n} \otimes \mathbb{C}_\mu^{\otimes n})^T \neq 0 && \text{for some } n \geq 1\end{aligned}$$

Set $\mathbb{L} = \mathcal{L}_\lambda \otimes \mathbb{C}_\mu$.
$H^0(X_w, \mathbb{L}^{\otimes n})^T \neq 0$ for some $n$ means $\mathbb{L}$ has semistable points on $X_w$.
The Hilbert-Mumford numerical criterion for semistability:

$$\mu^{\mathbb{L}}(x, \eta) \leq 0,$$

for every one-parameter subgroup $\eta : \mathbb{C}^* \to T$, where $\mu^{\mathbb{L}}(x, \eta)$ measures the action of $\mathbb{C}^*$ via $\eta$ on the fibre of $\mathbb{L}$ over the nearest fixed point to $x$.
If $x \in P(\eta)qB/B \cap X_w$, then

$$\mu^{\mathbb{L}}(x, \eta) = \langle \lambda, q^{-1}\eta \rangle - \langle \mu, \eta \rangle$$

# Inequalities from GIT

Thus $\mu \in P_\lambda^w$ if and only if

$$\langle \mu, \eta \rangle \geq \langle \lambda, q^{-1}\eta \rangle$$

for every OPS $\eta$ and $q \in W$.
That's too many inequalities.
It suffices to consider only those $q$ such that $P(\eta)qB/B \cap X_w$ is dense in $X_w$ (Hesselink).
It also suffices to consider only those $\eta$ which are extremal in their Weyl chamber: $\eta = v\omega_j^\vee$, where $\omega_j^\vee$ is a fundamental dominant coweight.

# Inequalities from GIT

## Theorem (Besson-Jeralds-K)

*$\mu \in P_\lambda^w$ if and only if, for all dominant fundamental coweights $\omega_j^\vee$ and $v \in W$,*

$$\langle \mu, v\omega_j^\vee \rangle \geq \langle \lambda, q^{-1} v\omega_j^\vee \rangle,$$

*where $vP_jv^{-1}qB/B \cap X_w$ is dense in $X_w$.*

The density criterion essentially boils down to a problem on the Bruhat decomposition: find the smallest $x$ such that

$$Bw^{-1}BvB \subseteq \overline{BxB} = \bigcup_{x' \leq x} Bx'B$$

Answer: $x = w^{-1} * v$. (Demazure product: $s_i * s_i = s_i$)

# Inequalities from GIT

**Theorem (Besson-Jeralds-K)**

*$\mu \in P_\lambda^w$ if and only if, for all dominant fundamental coweights $\omega_j^\vee$ and $v \in W$,*

$$\langle \mu, v\omega_j^\vee \rangle \geq \langle \lambda, w^{-1} * v\omega_j^\vee \rangle.$$

Interestingly enough,

$$w^{-1} * v = \max\{q^{-1}v | q \leq w\}.$$

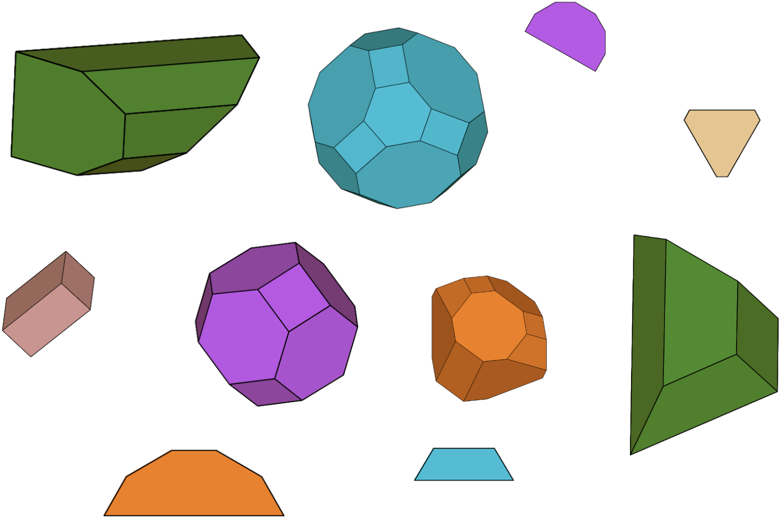
# Good Pairings

Did those inequalities help us?
In classical types ($A$, $B$, $C$, $D$), yes.
Remember we are trying to find $k$ such that $\mu = \mu_0 + k\alpha_i \in P_\lambda^{s_i w}$.
If $\mu$ lies on a face, we might need to divide by $\langle \alpha_i, v\omega_j^\vee \rangle$.
That is at most 2 for classical types.
There is always an integer between two distinct elements of $\frac{1}{2}\mathbb{Z}$.

**Theorem (Besson-Jeralds-K)**

*If $G$ has all its simple factors of types $A$, $B$, $C$, or $D$, then $\mathrm{char}(V_\lambda^w)$ is saturated for every choice of $\lambda$ and $w$.*

# Faces of Demazure Polytopes are Demazure Polytopes



# Affine Types

In affine types, we ask the same question.
Now our Schubert varieties $X_w$ are parametrized by the affine Weyl group $w \in W_{af}$.
Though they are defined inside the infinite-dimensional affine flag variety, they are still finite-dimensional.
The weight space question for $V_\lambda^w$ is still a GIT question for the finite-dimensional torus $\widehat{T}$ acting on $X_w$.

# 3 Families of Inequalities

We can still use GIT, but our one-parameter subgroups now come in three distinct flavors:

- $\eta$ is conjugate to a dominant weight, or
- $-\eta$ is conjugate to a dominant weight, or
- $\langle \delta, \eta \rangle = 0$

There will therefore be three different kinds of inequalities, due to the nature of deciding when

$$P(\eta)qI/I \cap X_w$$

is dense in $X_w$.

# 3 Families of Inequalities

First, we describe three Demazure actions. The monoid $(W_{af}, *)$ acts on the set $W_{af}$ in three different ways:

- *Usual Demazure action*: $w * v = \max\{qv | q \leq w\}$
- *Opposite action*: $w *_- v = \min\{qv | q \leq w\}$
- *Semiinfinite action*: $w *_{\frac{\infty}{2}} v = \max_{\leq_{\frac{\infty}{2}}}\{qv | q \leq w\}$

Note that $\min(S) = \max_{\leq_-}(S)$, where $\leq_-$ is the opposite Bruhat order in the opposite flag variety.

# 3 Families of Inequalities

## Theorem (Besson-Jeralds-Hong-K)

*$\mu \in P_\lambda^w$ if and only if, for all dominant fundamental coweights $\Lambda_j^\vee$ and $v \in W_{af}$,*

$$\langle \mu, v\Lambda_j^\vee \rangle \geq \langle \lambda, w^{-1} * v\Lambda_j^\vee \rangle,$$

*and, for all dominant fundamental coweights $\Lambda_j^\vee$ and $v \in W_{af}$,*

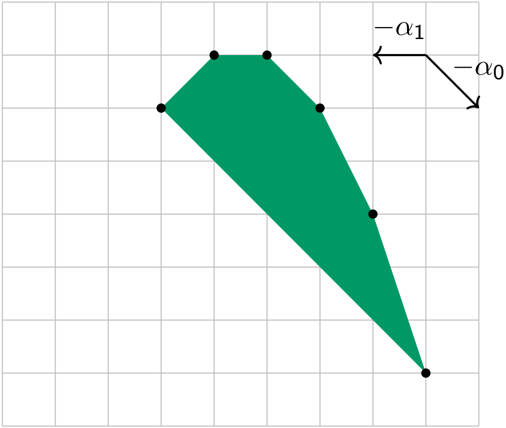
$$\langle \mu, v\Lambda_j^\vee \rangle \leq \langle \lambda, w^{-1} *_- v\Lambda_j^\vee \rangle,$$

*and, for all* finite *fundamental coweights $\omega_j^\vee$ and $v \in W_{fin}$,*

$$\langle \mu, v\omega_j^\vee \rangle \geq \langle \lambda, w^{-1} *_{\frac{\infty}{2}} v\omega_j^\vee \rangle.$$

# Example

Let $G = \widehat{\mathfrak{sl}}_2$, $\lambda = \Lambda_0 + \Lambda_1$.




The pairings $\langle \alpha_i, v\eta \rangle$ can now be arbitrarily high, so as of yet we cannot apply the same approach to the saturation question.

Thank You